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Non-collinearity in high energy processes

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Abstract. We discuss the treatment of intrinsic transverse momenta in high energy scattering processes. Within the field theoretical framework of QCD, the process is described in terms of correlators containing quark and gluon fields. The correlators, parametrized in terms of distribution and fragmentation functions, contain matrix elements of nonlocal field configurations requiring a careful treatment to assure colour gauge invariance. It leads to nontrivial gauge links connecting the parton fields. For the transverse momentum-dependent correlators the gauge links give rise to time reversal odd phenomena, showing up as single spin and azimuthal asymmetries. The gauge links, arising from multi-gluon initial and final state interactions, depend on the colour flow in the process, challenging universality.

Keywords. Partons; intrinsic transverse momentum; universality.

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1. Introduction

The basic degrees of freedom that feel the strong interactions, quarks and gluons, are confined into hadrons, strongly interacting particles. Considering the nucleons (light hadrons), the characteristic energy and distance scales are given by the nucleon mass $M_{\rm N}$, or taking into account the colour degrees of freedom one may prefer a scale $M_{\rm N}/N_{\rm c} \sim 300$ MeV. We refer to this as $\mathscr{O}(M)$ or $\mathscr{O}(Q^0)$ if we consider high-energy processes. Such processes are characterized by hard kinematical variables that are of order Q with $Q^2 \gg M_{\rm N}^2$. Depending on details, the high-energy scale Q can be the CM energy, $Q \sim \sqrt{s}$ or it can be a measure of the exchanged momentum.

The basic framework for the strong interactions is quantum chromodynamics (QCD). Hadrons, however, do not correspond to free particle states created via the quark and gluon operators in QCD. The situation thus differs from that of QED with physical electrons and photons. In the latter case, one knows how in the calculation of an S-matrix element contraction of annihilation and creation operator in the field and particle state leads to the spinor wave function. For positive times $\xi^0 = t$, one has

$$\langle 0|\psi_i(\xi)|p\rangle = \langle 0|\psi_i(\xi)\, b^\dagger(\mathbf{p})|0\rangle = \langle 0|\psi_i(0)|p\rangle\, \mathrm{e}^{-i\,p\cdot\xi} = u_i(\mathbf{p})\, \mathrm{e}^{-ip\cdot\xi}, \qquad (1)$$

with $p^0 = E_p = \sqrt{\mathbf{p}^2 + m^2}$. Such a matrix element is 'untruncated' as seen from

$$\langle 0|\psi_i(\xi)|p\rangle\,\theta(t) = \theta(t)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{-ik\cdot\xi}\,\frac{i(\not k + m)}{k^2 - m^2 + i\epsilon}$$
$$\times\frac{u_i(\mathbf{p})}{2m}\,(2\pi)^3\,2E_p\,\delta^3(\mathbf{k}-\mathbf{p}). \tag{2}$$

In a process involving a composite hadronic state $|P\rangle$, contractions with one or several of the quark and gluon operators may be involved, leading to nonzero matrix elements for a quark between the hadron state and a remainder, but also to nonzero matrix elements involving multi-parton field combinations,

$$\langle X|\psi_i(\xi)|P\rangle,\ \langle X|A^\mu(\eta)\,\psi(\xi)|P\rangle, \ldots\ .$$

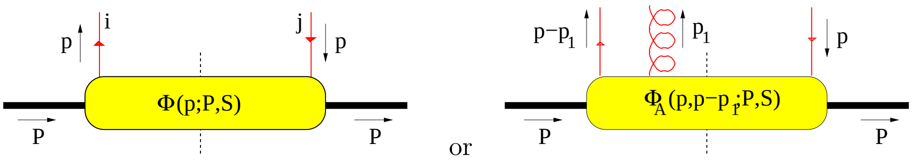
For a particular hadron and a parton field combination, one may collect those operators that involve hadron $|P\rangle$ into (distribution) correlators

$$\Phi_{ij}(p;P) = \sum_X \int \frac{\mathrm{d}^3P_X}{(2\pi)^3\,2E_X}\,\langle P|\bar\psi_j(0)|X\rangle\,\langle X|\psi_i(0)|P\rangle\,\delta^4(p+P_X-P)$$
$$= \frac{1}{(2\pi)^4}\int \mathrm{d}^4\xi\ \mathrm{e}^{i\,p\cdot\xi}\,\langle P|\bar\psi_j(0)\,\psi_i(\xi)|P\rangle, \tag{3}$$

or correlators involving matrix elements of the form

$$\Phi^\mu_{ij}(p,p_1;P) = \frac{1}{(2\pi)^8}\int \mathrm{d}^4\xi\,\mathrm{d}^4\eta\ \mathrm{e}^{i\,(p-p_1)\cdot\xi}$$
$$\times\mathrm{e}^{i\,p_1\cdot\eta}\,\langle P|\bar\psi_j(0)\,A^\mu(\eta)\,\psi_i(\xi)|P\rangle, \tag{4}$$

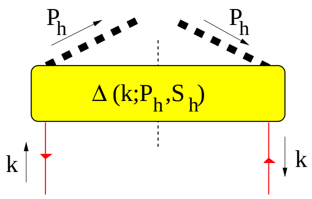
pictorially,



We will not attempt to calculate these, but leave them as the soft parts, requiring nonperturbative QCD methods to calculate them. In particular, although being 'untruncated' in the quark legs, they will no longer exhibit poles corresponding to free quarks. These are fully unintegrated parton correlators for initial state hadrons, in general quite problematic quantities. For example, they are by themselves not even colour gauge-invariant, an issue to be discussed below. When more hadrons are involved, one could consider two-hadron correlators, involving two-hadron states (or correlators involving hadronic states in initial and final state), etc. If the hadrons are well-separated in momentum phase-space with $P_i \cdot P_j \sim Q^2$, one expects on dimensional grounds that incoherent contributions are suppressed by $1/(P_i - P_j)^2 \sim 1/Q^2$ and one can (at least naively) factorize using forward correlators for single hadrons, connected by a hard partonic subprocess. Such a separation in momentum space requires a hard inclusive scattering process ($Q^2 \sim s$). The inclusive character is needed to assure that partons originate from one hadron, leaving a (target) jet. In turn, final state partons decay into a jet, in which we also consider a single identified hadron, which can straightforwardly be extended to a multi-particle, e.g. two-pion, state. For the fragmentation process of a parton (with momentum k) into hadrons (with momentum P_h) we combine the decay matrix elements in the (fragmentation) correlator, for quarks

$$\begin{aligned}\Delta_{ij}(k,P_h) &= \sum_X \frac{1}{(2\pi)^4}\int \mathrm{d}^4\xi\ \mathrm{e}^{ik\cdot\xi}\,\langle 0|\psi_i(\xi)|P_h,X\rangle\langle P_h,X|\bar{\psi}_j(0)|0\rangle \\ &= \frac{1}{(2\pi)^4}\int \mathrm{d}^4\xi\ \mathrm{e}^{ik\cdot\xi}\,\langle 0|\psi_i(\xi)a_h^\dagger a_h\bar{\psi}_j(0)|0\rangle, \end{aligned} \tag{5}$$

where an averaging over colour indices is implicit. Pictorially we have



In particular, we note that in fragmentation correlators one no longer deals with plane-wave hadronic states, but with out-states $|P_h, X\rangle$. In all of the hadronic states mentioned before, one can also consider polarized hadronic states. The spin of quarks is contained in Dirac structure and that of gluons in the Lorentz structure of correlators.

The basic idea in the diagrammatic approach is to realize that the correlator involves both hadronic states and quark and gluon operators. The correlators can be studied independent of the hard process, provided we have dealt with the issue of colour gauge invariance. The correlator is the Fourier transform in the space-time arguments of the quark and gluon fields. In the correlators, all momenta of hadrons and quarks and gluons (partons) inside the hadrons are soft which means that $p^2 \sim p\cdot P \sim P^2 = M_{\mathrm{N}}^2 \ll Q^2 \sim s$. The off-shellness being of hadronic order implies that in the hard process partons are in essence on-shell. Consistency of this may be checked using QCD interactions to give partons a large off-shellness of $\mathcal{O}(Q)$ and check the behaviour as a function of the momenta. In these considerations one must also realize that beyond tree-level one has to distinguish bare and renormalized fields.

2. Collinear and transverse momentum-dependent correlators

In a hard process, the parton fields that appear in different correlators correspond to partons in the subprocess for which the momenta satisfy $p_i \cdot p_j \sim Q^2$. In the study of a particular correlator it implies the presence of a 'hard' environment. To connect the correlator to the hard part of the process, it is useful to introduce for each correlator with hadron momentum P, a null-vector n, such that $P\cdot n \sim Q$. Using this relation, n would be dimensionless. It is actually more convenient to replace $n/(P\cdot n)$ by a dimensionful null-vector $n \sim 1/Q$, such that $P\cdot n = 1$. The vectors P and n can be used to keep track of the importance of various terms in the correlators and in the components of momentum and spin vectors [1a]. The n-vector will acquire a meaning in the explicit applications or play an intermediary role. At leading order, it will turn out that the precise form of n does not matter, but at subleading $(1/Q)$ order one needs to be careful.

For parton momenta we write the Sudakov decomposition

$$p = x\,P + p_{\mathrm{T}} + \underbrace{(p\cdot P - x\,M^2)}_{\sigma}\,n, \tag{6}$$

where the term $x\,P \sim Q$, $p_{\mathrm{T}} \sim M$ and $\sigma\,n \sim M^2/Q$. We have the exact relations $p\cdot p_{\mathrm{T}} = p_{\mathrm{T}}^2 = (p - x\,P)^2$. The momentum fraction $x = p\cdot n$ is $\mathscr{O}(1)$.

In a hard process, the importance of the various components allows up to specific orders in $1/Q$, an integration over some components of the parton momenta. The fact that the main contribution in $\Phi(p;P)$ is assumed to come from regions where $p\cdot P \le M^2$, whereas the momenta have characteristic scale Q, allows performing the σ-integration up to M^2/Q^2 contributions (and possible contributions from nonintegrable tails). The resulting transverse momentum-dependent (TMD) correlators are light-front correlators,

$$\begin{aligned}\Phi_{ij}(x, p_{\mathrm{T}}; n) &= \int \mathrm{d}(p\cdot P)\Phi_{ij}(p;P) \\ &= \int \frac{\mathrm{d}(\xi\cdot P)\,\mathrm{d}^2\xi_{\mathrm{T}}}{(2\pi)^3}\;\mathrm{e}^{i\,p\cdot\xi}\;\langle P|\bar{\psi}_j(0)\,\psi_i(\xi)|P\rangle\Big|_{\mathrm{LF}},\end{aligned} \tag{7}$$

where we have suppressed the dependence on hadron momentum P. The subscript LF refers to light-front, implying $\xi\cdot n = 0$. The light-cone correlators are the correlators containing the parton distribution functions depending only on the light-cone momentum fraction x,

$$\begin{aligned}\Phi_{ij}(x; n) &= \int \mathrm{d}(p\cdot P)\,\mathrm{d}^2p_{\mathrm{T}}\Phi_{ij}(p;P) \\ &= \int \frac{\mathrm{d}(\xi\cdot P)}{(2\pi)}\;\mathrm{e}^{i\,p\cdot\xi}\;\langle P|\bar{\psi}_j(0)\,\psi_i(\xi)|P\rangle\Big|_{\mathrm{LC}},\end{aligned} \tag{8}$$

where the subscript LC refers to light-cone, implying $\xi\cdot n = \xi_{\mathrm{T}} = 0$. This integration is generally allowed (again up to M^2/Q^2 contributions and contributions coming from tails, e.g. logarithmic corrections from $1/p_{\mathrm{T}}^2$ tails) if we are interested in hard processes, in which only hard scales (large invariants $\sim Q^2$ or ratios thereof, angles, rapidities) are measured. If one considers hadronic scale observables (correlations or transverse momenta in jets, slightly off-collinear configurations) one will need the TMD correlators.

The correlators encompass the information on the soft parts. They depend on the hadron and (contained) quark momenta P and p (and spin vectors). The structure of the correlator is reproduced from these momenta incorporating the required Dirac and Lorentz structure. Clearly, it is advantageous to maximize the number of components along the momentum (collinear). For the soft scalar objects this means maximizing the number of contractions with n. This leads for nonlocal operators to the dominance of the twist-2 operators

$$\bar{\psi}(0)\not{n}\psi(\xi) \quad \text{and} \quad F^{n\alpha}F^{n\beta}(\xi) \tag{9}$$

(the latter with transverse indices α and β). Twist in this case is just equal to the canonical dimension of the operator combination (remember that $\dim(n) = -1$).

Of course the appearance of the field strength tensor rather than the gauge field is a requirement of gauge invariance. Besides the field tensor, we need the inclusion of gauge links

$$U^{[n]}_{[0,\xi]} = \mathscr{P} \exp\left(-i\int_0^\xi \mathrm{d}(\eta\cdot P)\, n\cdot A(\eta)\right), \tag{10}$$

connecting coloured parton fields. In the case of the collinear correlators, the gauge links can be built from the $\mathscr{O}(1)$ gauge fields $A^+ = A^n = n\cdot A$, giving a link along the light-cone ($\xi^+ = n\cdot\xi = \xi_{\mathrm{T}} = 0$). The colour gauge-invariant light-cone correlators for quarks and gluons are

$$\Phi_{ij}(x;n) = \int \frac{\mathrm{d}(\xi\cdot P)}{(2\pi)} \mathrm{e}^{i\,p\cdot\xi} \langle P|\bar{\psi}_j(0)\, U^{[n]}_{[0,\xi]}\, \psi_i(\xi)|P\rangle\Big|_{\mathrm{LC}}, \tag{11}$$

$$\Gamma^{\alpha\beta}(x;n) = \int \frac{\mathrm{d}(\xi\cdot P)}{(2\pi)} \mathrm{e}^{i\,p\cdot\xi} \langle P|\mathrm{Tr}\left(F^{n\beta}(0)\, U^{[n]}_{[0,\xi]}\, F^{n\alpha}(\xi)\, U^{[n]}_{[\xi,0]}\right)|P\rangle\Big|_{\mathrm{LC}}. \tag{12}$$

Using the Taylor expansion of the colour gauge-invariant nonlocal operators,

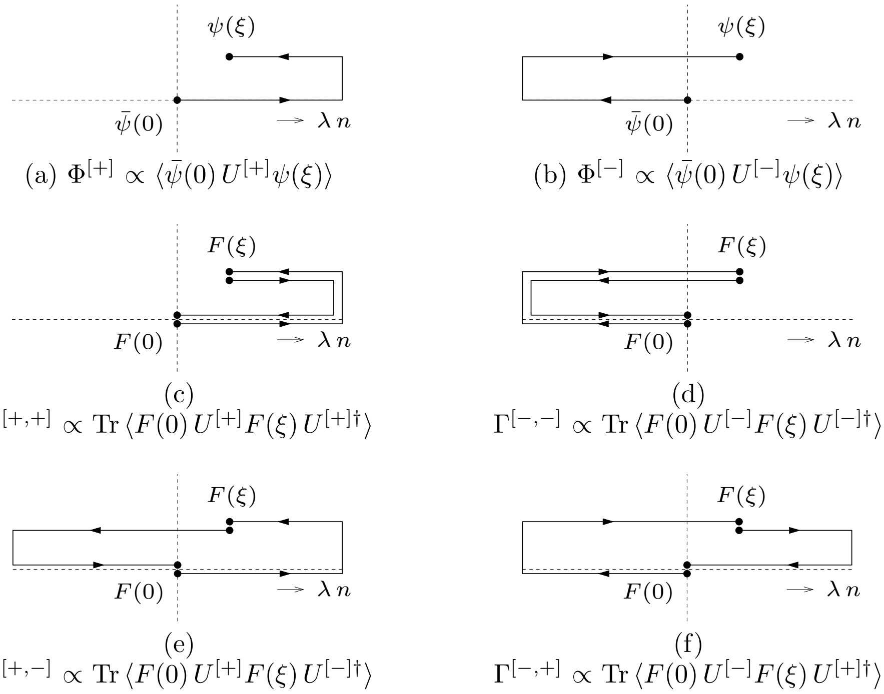
$$\psi^\dagger(0)\, U_{[0,\xi]}\psi(\xi) = \sum_{n=0}^{\infty}(-i)^n\, \frac{x_{\mu_1}\ldots x_{\mu_n}}{n!}\, \psi^\dagger(0)\, iD^{\mu_1}\ldots iD^{\mu_n}\, \psi(0)$$

one recovers the irreducible set of symmetric traceless local operators relevant in the operator product expansion (OPE) approach to describe $\Phi_{ij}(x)$ and $\Gamma_{\alpha\beta}(x)$, namely

$$O^{\mu_1\ldots\mu_n}_{\mathrm{quarks}\,ij} = \bar{\psi}_j(0)\,\gamma^{\{\mu_1}\, iD^{\mu_2}\ldots iD^{\mu_n\}}\,\psi_i(0) - \text{traces},$$
$$O^{\mu_1\ldots\mu_n}_{\mathrm{gluons}\,\alpha\beta} = -F_\beta^{\ \{\mu_1}(0)\, iD^{\mu_2}\ldots iD^{\mu_{n-1}}\, F^{\mu_n\}}_{\ \ \ \alpha}(0) - \text{traces},$$

in which the spin n represents the number of symmetrized indices. Subtracting traces is needed to have an irreducible set. The TMD light-front correlators

$$\Phi^{[C]}_{ij}(x,p_{\mathrm{T}};n) = \int \frac{\mathrm{d}(\xi\cdot P)\,\mathrm{d}^2\xi_{\mathrm{T}}}{(2\pi)^3}\ \mathrm{e}^{i\,p\cdot\xi}\ \langle P|\bar{\psi}_j(0)\, U^{[n,C]}_{[0,\xi]}\, \psi_i(\xi)|P\rangle\Big|_{\mathrm{LF}}, \tag{13}$$



(a) $\Phi^{[+]} \propto \langle \bar{\psi}(0)\, U^{[+]} \psi(\xi) \rangle$

(b) $\Phi^{[-]} \propto \langle \bar{\psi}(0)\, U^{[-]} \psi(\xi) \rangle$

(c) $\Gamma^{[+,+]} \propto \mathrm{Tr}\, \langle F(0)\, U^{[+]} F(\xi)\, U^{[+]\dagger} \rangle$

(d) $\Gamma^{[-,-]} \propto \mathrm{Tr}\, \langle F(0)\, U^{[-]} F(\xi)\, U^{[-]\dagger} \rangle$

(e) $\Gamma^{[+,-]} \propto \mathrm{Tr}\, \langle F(0)\, U^{[+]} F(\xi)\, U^{[-]\dagger} \rangle$

(f) $\Gamma^{[-,+]} \propto \mathrm{Tr}\, \langle F(0)\, U^{[-]} F(\xi)\, U^{[+]\dagger} \rangle$

Figure 1. Simplest structures (without loops) for gauge links and operators in quark correlators (**a**)–(**b**) and gluon correlators (**c**)–(**f**).

$$
\begin{aligned}
\Gamma_{\alpha\beta}^{[C,C']}(x, p_{\mathrm{T}}; n) = & \int \frac{\mathrm{d}(\xi \cdot P)\, \mathrm{d}^2\xi_{\mathrm{T}}}{(2\pi)^3}\ \mathrm{e}^{i\,p\cdot\xi} \\
& \times \langle P | \mathrm{Tr} \left(F_\beta^n(0)\, U_{[0,\xi]}^{[n,C]}\, F_\alpha^n(\xi)\, U_{[\xi,0]}^{[n,C']} \right) | P \rangle \Big|_{\mathrm{LF}}, \qquad (14)
\end{aligned}
$$

involve a more complex link structure, leading to a path dependence in the definitions (indicated by the arguments C and C'). This arises because of the (necessary) transverse piece(s) in the gauge link. The simplest possibilities for the links in the case of quark and gluon correlators are shown in figure 1 [1].

3. The observables

An important aspect of incorporating intrinsic transverse momenta is the possibility to access them in experiments. Consider the subprocess $\gamma^*(q) + q(p) \to q(k)$ which dominates at leading order the inclusive deep inelastic scattering (DIS) process $\gamma^*(q) + N(P) \to X$. In collinear approximation ($p \approx xP$) one finds for the momentum fraction x the well-known relation $x = x_B = Q^2/2P \cdot q$, i.e. the fraction is identified with the Bjorken scaling variable. For the semi-inclusive deep inelastic scattering (SIDIS) process $\gamma^*(q) + N(P) \to h(P_h) + X$ with the same subprocess

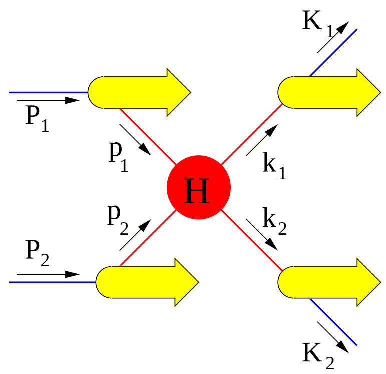


Figure 2. Schematic illustration of the contribution of a hard subprocess, parton (p_1) + parton (p_2) $\rightarrow$ parton (k_1) + parton (k_2), to the (2-particle inclusive) scattering process hadron (P_1) + hadron (P_2) $\rightarrow$ hadron (K_1) + hadron (K_2) + X, at the level of the amplitude. The process being hard implies for the hadronic momenta $P_1 \cdot P_2 \sim P_1 \cdot K_1 \sim Q^2$, etc.

one has in collinear approximation ($k \approx P_h/z$), the relation $z = z_h = P \cdot P_h / P \cdot q$. The non-collinearity in this process is determined by

$$q_{\mathrm{T}} = q + x_B\, P - \frac{1}{z_h}\, P_h, \tag{15}$$

which is taken as zero in collinear approximation. First of all, we note that q_{T} is experimentally measurable being a difference of vectors of $\mathcal{O}(Q)$. This difference is meaningful at $\mathcal{O}(\mathcal{M})$ or higher because mass corrections, coming from the identification of x with x_B and z with z_h, appear only at $\mathcal{O}(1/Q^2)$. The vector q_{T} is the transverse momentum of q in a frame in which P and P_h are chosen parallel or (experimentally more useful) it is the transverse momentum $-P_{h\perp}/z_h$ in a frame in which q and P are chosen parallel. We write $Q_{\mathrm{T}}^2 = -q_{\mathrm{T}}^2$. When $Q_{\mathrm{T}} \sim \mathcal{O}(M)$ one easily sees that the Sudakov expansions for the quark momenta, $p \approx xP + p_{\mathrm{T}}$ and $k \approx \frac{1}{z}\, P_h + k_{\mathrm{T}}$, imply that $q_{\mathrm{T}} = k_{\mathrm{T}} - p_{\mathrm{T}}$ and the q_{T}-dependence is attributed to the (convoluted effect of the) intrinsic transverse momenta in the fragmentation and distribution correlators. When $Q_{\mathrm{T}} \sim \mathcal{O}(Q)$ a collinear description involving a subprocess with one additional parton radiated off is needed, but for consistency one also wants a match with the TMD description [2].

Not only in electroweak processes like SIDIS or the Drell–Yan process transverse momenta can be accessed. This can also be achieved for hadron–hadron scattering. Also here the identification of the transverse momentum is only possible together with the identification of the hard subprocess (as shown in figure 2). We define

$$q_{\mathrm{T}} = \frac{1}{z_1}\, K_1 + \frac{1}{z_2}\, K_2 - x_1\, P_1 - x_2\, P_2 = p_{1\mathrm{T}} + p_{2\mathrm{T}} - k_{1\mathrm{T}} - k_{2\mathrm{T}}, \tag{16}$$

a relation valid up to $\mathcal{O}(M)$. The momenta involved to find q_{T} are in principle all $\mathcal{O}(Q)$ and using at leading order $q_{\mathrm{T}} \approx 0$ yields relations for the momentum fractions in terms of the external hadron momenta (up to $1/Q^2$ corrections). The determination of $q_{\rm T}$ at $\mathscr{O}(M)$ gives access to the transverse momenta. Experimentally one component of $q_{\rm T}$ is found as the non-collinearity of the produced particles K_1 and K_2 in the plane perpendicular to the colliding particles P_1 and P_2, outlined in detail in [3].

Accessing intrinsic transverse momenta in most cases requires a careful study of azimuthal dependence in high energy processes. Although the effects are in principle not suppressed by powers of the hard scale in comparison with the leading collinear treatment, it requires measuring hadronic scale quantities (transverse momenta) in a high momentum environment. Symmetries, in particular time reversal (T) invariance play an important role:

- The theory of QCD is T-invariant. This makes it sensible to distinguish quantities and observables according to their T-behaviour.
- For distribution correlators involving plane-wave hadronic states in the definition, combination of the T-operation and hermiticity, shows that the collinear correlators $\Phi(x)$ and $\Gamma(x)$ must be T-even. For the TMD correlators, however, the T-operation interchanges $\Phi^{[+]}(x, p_{\rm T}) \leftrightarrow \Phi^{[-]}(x, p_{\rm T})$ (and similar relations for gluon TMD correlators). This allows to construct T-even and T-odd combinations.
- For fragmentation functions the appearance of a hadronic out-state in the definition, prohibits the use of T-symmetry as a constraint and one has always both T-even and T-odd parts in the correlator (one can refer to T-even or T-odd in as far as the operator structure is concerned, referred to as naive T-even or naive T-odd).
- In a scattering process, in which T-symmetry can be used as a constraint, single spin asymmetries would be forbidden. In fact the only real example of this is DIS (omitting electromagnetic interaction effects). For hadron–hadron scattering, e.g. the Drell–Yan process, one has a two-hadron initial state and only the assumption of a factorized description would imply the absence of single spin asymmetries. We now know that this assumption is not valid, even not at leading order! Similarly, for processes with identified hadrons in the final state T-invariance does not give constraints.
- At leading order in α_s, however, it is possible to connect single spin asymmetries (T-odd observable) to the T-odd soft parts, since the hard process will be T-even at this leading order. Collins and Sivers effects as explanation for single spin asymmetries are the best-known examples.

4. The TMD master formula

The description of a hard process is obtained by writing down hard processes involving quarks and gluons and connecting these to the soft parts corresponding to initial state hadrons and observed hadrons in the final state. In the region where the hadrons are separated far enough in phase space ($P_i \cdot P_j \sim Q^2$, as discussed in the Introduction) one can have a soft part for each of the hadrons. For the determination of the relevant multi-parton matrix elements that need to be included in the calculation one can use the twist analysis alluded to in §2. At leading order one needs the leading twist matrix elements $\langle\bar{\psi}\psi\rangle$ and $\langle A_{\rm T} A_{\rm T}\rangle$, but also the multi-parton matrix elements $\langle\bar{\psi}\,A^+\ldots A^+\psi\rangle$ and $\langle A_{\rm T}\,A^+\ldots A^+A_{\rm T}\rangle$ (all having the same twist). The various matrix elements are resummed into colour-gauge invariants combinations. Inclusion of the transverse pieces at infinity requires a careful analysis [4]. The links that arise are process-dependent. They arise from diagrammatic contributions where collinear gluons $A\cdot n$ belonging to a particular soft part are attached to parton lines belonging to a different soft part (which are precisely the external parton lines of the linking hard subprocess). The link structure, thus, is not affected by inclusion of QCD corrections. On the other hand, the link structure depends on the colour-flow in the specific diagram.

The resulting expression for a hard cross-section at measured $q_{\rm T}$ is

$$\frac{{\rm d}\sigma}{{\rm d}^2q_{\rm T}} \sim \sum_{D,abc\ldots} \Phi_a^{[C_1(D)]}(x_1,p_{1{\rm T}})\,\Phi_b^{[C_2(D)]} \\ \times(x_2,p_{2{\rm T}})\,\hat{\sigma}^{[D]}_{ab\to c\ldots}\Delta_c^{C_1'(D)]}(z_1,k_{1{\rm T}})\cdots+\cdots, \tag{17}$$

where the sum D runs over diagrams distinguishing also the particular colour flow and $abc\ldots$ is the summation over quark and antiquark flavours and gluons. All Dirac and Lorentz indices, traces, etc. are suppressed. The ellipsis at the end indicate contributions of other hard processes.

We illustrate this master formula in the following example, taken from [5], describing the contribution in a hard scattering process coming from the $qq\to qq$ subprocess (with both quarks having the same flavour). There are four diagrammatic contributions (see table 1), the result of which can be denoted as $\hat{\sigma}^{[D]}_{qq\to qq}$ with D running over the diagrams. For the first diagram, there are two different possibilities for the colour flow, which absorbing the overall colour factor in $\hat{\sigma}$ have strengths $(N_{\rm c}^2+1)/(N_{\rm c}^2-1)$ and $-2/(N_{\rm c}^2-1)$, respectively. The second diagram has the same colour flow possibilities. The third and fourth diagrams also have identical colour flow possibilities but different from the first two diagrams. In this case each of the diagrams contributes two terms to the sum in eq. (17), e.g. the first diagram yields

$$\frac{{\rm d}\sigma}{{\rm d}^2q_{\rm T}} \sim \Phi_q^{[(\Box)+]}(1)\,\Phi_q^{[(\Box)+]}(2)\,\underbrace{\frac{N_{\rm c}^2+1}{N_{\rm c}^2-1}\,\hat{\sigma}^{[D_1]}_{qq\to qq}}\,\Delta_q^{[(\Box)-^\dagger]}(1')\,\Delta_q^{[(\Box)-^\dagger]}(2') \\ +\Phi_q^{[\Box+]}(1)\,\Phi_q^{[\Box+]}(2)\,\underbrace{\frac{-2}{N_{\rm c}^2-1}\,\hat{\sigma}^{[D_1]}_{qq\to qq}}\,\Delta_q^{[\Box-^\dagger]}(1')\,\Delta_q^{[\Box-^\dagger]}(2')+\cdots, \tag{18}$$

where the underbraced items are separate $\hat{\sigma}^{[D]}$-entries in the D-summation of eq. (17).

5. Integrated and weighted cross-section

The results for cross-sections after integration over the transverse momenta $q_{\rm T}$ involve the path-independent integrated correlators $\Phi(x)$ rather than the path-dependent TMD correlators $\Phi^{[C(D)]}(x,p_{\rm T})$. Thus, from eq. (17) one gets the well-known result

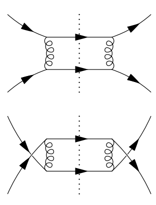
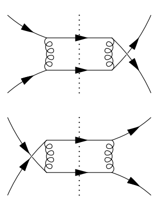
	$\Phi_q \propto \langle\, \overline{\psi}(0) \left\{ \frac{N_c^2+1}{N_c^2-1} \frac{\mathrm{Tr}\left[\mathcal{U}^{[\Box]}\right]}{N_c} \mathcal{U}^{[+]} - \frac{2}{N_c^2-1} \mathcal{U}^{[\Box]}\mathcal{U}^{[+]} \right\} \psi(\xi) \,\rangle$ $\Delta_q \propto \langle\, \overline{\psi}(\xi) \left\{ \frac{N_c^2+1}{N_c^2-1} \frac{\mathrm{Tr}\left[\mathcal{U}^{[\Box]}\right]}{N_c} \mathcal{U}^{[-]\dagger} - \frac{2}{N_c^2-1} \mathcal{U}^{[\Box]}\mathcal{U}^{[-]\dagger} \right\} \psi(0) \,\rangle$
	$\Phi_q \propto \langle\, \overline{\psi}(0) \left\{ \frac{2N_c^2}{N_c^2-1} \frac{\mathrm{Tr}\left[\mathcal{U}^{[\Box]}\right]}{N_c} \mathcal{U}^{[+]} - \frac{N_c^2+1}{N_c^2-1} \mathcal{U}^{[\Box]}\mathcal{U}^{[+]} \right\} \psi(\xi) \,\rangle$ $\Delta_q \propto \langle\, \overline{\psi}(\xi) \left\{ \frac{2N_c^2}{N_c^2-1} \frac{\mathrm{Tr}\left[\mathcal{U}^{[\Box]}\right]}{N_c} \mathcal{U}^{[-]\dagger} - \frac{N_c^2+1}{N_c^2-1} \mathcal{U}^{[\Box]}\mathcal{U}^{[-]\dagger} \right\} \psi(0) \,\rangle$

Table 1. Gauge-links appearing in the soft parts connected to the $qq \to qq$ subprocess depend on the specific diagrams. The paths for $\Phi_q(x, p_T)$ are shown in figure 3.

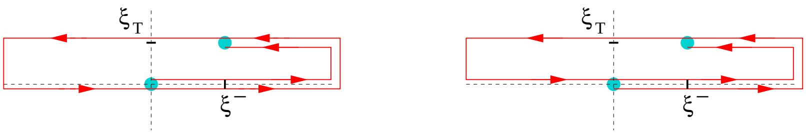


Figure 3. The paths in the gauge links in $\Phi_q(x, p_T)$. They involve a loop $U^{[\Box]} = U^{[+]}\, U^{[-]\dagger}$, which in the path shown in the left figure is closed (colour-trace) and in the right figure is followed by a [+]-path. We will use the short-hand notations $\Phi^{[(\Box)+]}$ and $\Phi^{[\Box+]}$ respectively with $(\Box)$ indicating the colour-tracing and averaging.

$$\sigma \sim \sum_{abc\ldots} \Phi_a(x_1)\, \Phi_b(x_2)\, \hat{\sigma}_{ab\to c\ldots} \Delta_c(z_1) \cdots + \cdots, \tag{19}$$

where

$$\hat{\sigma}_{ab\to c\ldots} = \sum_D \hat{\sigma}^{[D]}_{ab\to c\ldots} \tag{20}$$

is the partonic cross-section.

Constructing a weighted cross-section (azimuthal asymmetry) by including a weight q_T^α in the q_T-integration leads, with the help of the relation between the observable q_T and the intrinsic transverse momenta (e.g. the relation $q_T = p_T - k_T$ in SIDIS), to soft correlators of the form

$$\Phi_\partial^{\alpha\,[C]}(x) = \int \mathrm{d}^2 p_T\; p_T^\alpha \Phi^{[C]}(x, p_T). \tag{21}$$

These still contain a path dependence, and so eq. (17) cannot be simplified immediately. However, it turns out that the correlator in eq. (21) can be expressed as

$$\Phi_{\partial}^{\alpha\,[C]}(x) = \widetilde{\Phi}_{\partial}^{\alpha}(x) + C_G^{[U(C)]}\,\pi\Phi_G^{\alpha}(x,x). \tag{22}$$

Here $\widetilde{\Phi}_{\partial}(x)$ is a collinear correlator containing matrix elements with T-even operators, while $\Phi_G(x, x_1)$ is a collinear correlator with a structure like the quark–gluon–quark correlator shown in eq. (4) involving the gluon field $F^{n\alpha}$. In eq. (22) one needs the zero-momentum ($x_1 = 0$) limit for the gluon momentum. This matrix element is known as the gluonic pole matrix element. The operators involved are T-odd. Both collinear correlators on the RHS in eq. (22) are link-independent. The gluonic pole factors C_G multiplying the gluonic pole correlator in eq. (22), however, do depend on the gauge link. They can be easily calculated. We have for instance $C_G^{[\pm]} = \pm 1$, $C_G^{[\Box +]} = 3$ and $C_G^{[(\Box)+]} = 1$. Thus, one can write for the single-weighted cross-section

$$\begin{aligned}
\langle q_{\mathrm{T}}^{\alpha}\sigma\rangle &= \int \mathrm{d}^2 q_{\mathrm{T}}\, q_{\mathrm{T}}^{\alpha} \frac{\mathrm{d}^2\sigma}{\mathrm{d}^2 q_{\mathrm{T}}} \\
&= \sum_{D,abc\ldots} \Phi_{\partial\,a}^{\alpha\,[C]}(x_1)\,\Phi_b(x_2)\,\hat{\sigma}_{ab\to c\ldots}^{[D]}\Delta_{\mathrm{c}}(z_1)\cdots + \cdots \\
&= \sum_{abc\ldots} \widetilde{\Phi}_{\partial\,a}^{\alpha}(x_1)\,\Phi_b(x_2)\hat{\sigma}_{ab\to c\ldots}\Delta_{\mathrm{c}}(z_1)\cdots + \cdots \\
&\quad + \sum_{abc\ldots} \pi\Phi_{G\,a}^{\alpha}(x_1,x_1)\,\Phi_b(x_2)\,\hat{\sigma}_{[a]b\to c\ldots}\Delta_{\mathrm{c}}(z_1)\cdots + \cdots,
\end{aligned} \tag{23}$$

where the first term is multiplied by the normal parton cross-section (eq. (20)) and the second one involves the gluonic pole cross-section,

$$\hat{\sigma}_{[a]b\to c\ldots} = \sum_D C_G^{[U(C(D))]}\hat{\sigma}_{ab\to c\ldots}^{[D]}\ . \tag{24}$$

Noteworthy is the fact that these gluonic pole cross-sections like the normal partonic cross-sections also constitute gauge invariant combinations of the squared amplitudes. While for the electroweak processes like SIDIS and DY one has a simple factor, $\hat{\sigma}_{\ell[q]\to\ell q} = +\hat{\sigma}_{\ell q\to\ell q}$ and $\hat{\sigma}_{[q]\bar{q}\to\ell\bar{\ell}} = -\hat{\sigma}_{q\bar{q}\to\ell\bar{\ell}}$, the result for $qq \to qq$ is more complex,

$$\hat{\sigma}_{qq\to qq} = \hat{\sigma}^{[D_1]} + \hat{\sigma}^{[D_2]} + \hat{\sigma}^{[D_3]} + \hat{\sigma}^{[D_4]}, \tag{25}$$

$$\hat{\sigma}_{[q]q\to qq} = \frac{N_{\mathrm{c}}^2 - 5}{N_{\mathrm{c}}^2 - 1}(\hat{\sigma}^{[D_1]} + \hat{\sigma}^{[D_2]}) - \frac{N_{\mathrm{c}}^2 + 3}{N_{\mathrm{c}}^2 - 1}(\hat{\sigma}^{[D_3]} + \hat{\sigma}^{[D_4]}), \tag{26}$$

where $\hat{\sigma}^{[D_i]}$ refer to the contributions coming from the diagrams in table 1. Actually the results simplify in the limit $N_{\mathrm{c}} \to \infty$ in which case the colour flow is unique for each diagram. Explicit results for gluonic pole cross-sections are given in ref. [8].

The approach to understand T-odd observables like single spin asymmetries via the TMD correlators and the non-trivial gauge link structure unifies a number of approaches to understand such observables, in particular the collinear approach of Qiu and Sterman [6] and the inclusion of soft gluon interactions by Brodsky and collaborators [7]. Although the treatment of fragmentation correlators also separates into parts with T-even and T-odd operator structure, gluonic pole contributions (T-odd parts) in the case of fragmentation might vanish. Indications come from the soft-gluon approach [9] and a recent spectral analysis in a spectator model approach [10].

6. Universality

Clearly the TMD master formula in eq. (17) breaks universality in the sense that one needs to know TMD correlators $\Phi^{[U]}(x, p_{\mathrm{T}})$ with all sorts of gauge links U. Certainly it would be desirable to perform a study of the effects on the soft correlators caused by more complex gauge links than the simple ones given in figure 1.

A useful procedure is to rewrite the TMD correlators in terms of T-even and T-odd correlators constructed from those in figure 1 and a residual or junk part,

$$\Phi^{[U]}(x, p_{\mathrm{T}}) = \Phi^{[\mathrm{even}]} + C_G^{[U]}\Phi^{[\mathrm{odd}]}(x, p_{\mathrm{T}}) + \delta\Phi^{[U]}(x, p_{\mathrm{T}}), \tag{27}$$

which by construction leads to

$$\Phi^{[\mathrm{even}]}(x) = \Phi(x), \quad \Phi_\partial^{\alpha\,[\mathrm{even}]}(x) = \widetilde{\Phi}_\partial^\alpha(x) \tag{28}$$

$$\Phi^{[\mathrm{odd}]}(x) = 0, \qquad \Phi_\partial^{\alpha\,[\mathrm{odd}]}(x) = \pi\,\Phi_G^\alpha(x, x) \tag{29}$$

$$\delta\Phi^{[U]}(x) = 0, \qquad \delta\Phi_\partial^{\alpha\,[U]}(x) = 0. \tag{30}$$

For the simple quark correlators one has $\delta\Phi^{[+]} = \delta\Phi^{[-]} = 0$ and the T-even and T-odd combinations are

$$\Phi^{[\mathrm{even}]}(x, p_{\mathrm{T}}) = \frac{1}{2}(\Phi^{[+]}(x, p_{\mathrm{T}}) + \Phi^{[-]}(x, p_{\mathrm{T}})) \tag{31}$$

$$\Phi^{[\mathrm{odd}]}(x, p_{\mathrm{T}}) = \frac{1}{2}(\Phi^{[+]}(x, p_{\mathrm{T}}) - \Phi^{[-]}(x, p_{\mathrm{T}})). \tag{32}$$

Then one finds for instance

$$\begin{aligned}\Phi^{[\Box+]}(x, p_{\mathrm{T}}) &= \Phi^{[\mathrm{even}]}(x, p_{\mathrm{T}}) + 3\Phi^{[\mathrm{odd}]}(x, p_{\mathrm{T}}) + \delta\Phi^{[\Box+]}(x, p_{\mathrm{T}}) \\ &= 2\,\Phi^{[+]}(x, p_{\mathrm{T}}) - \Phi^{[-]}(x, p_{\mathrm{T}}) + \delta\Phi^{[\Box+]}(x, p_{\mathrm{T}}).\end{aligned}$$

In [1] it was noted that some further nontrivial simplifications occur for quark junk TMD while also the simple gluon correlators (figure 1) can be regrouped into T-even and T-odd combinations,

$$\Gamma^{[\mathrm{even}]}(x, p_{\mathrm{T}}) = \frac{1}{2}\,\Gamma^{[+,+]}(x, p_{\mathrm{T}}) + \frac{1}{2}\,\Gamma^{[-,-]}(x, p_{\mathrm{T}}), \tag{33}$$

$$\Gamma_F^{[\mathrm{odd}]}(x, p_{\mathrm{T}}) = \frac{1}{2}\,\Gamma^{[+,+]}(x, p_{\mathrm{T}}) - \frac{1}{2}\,\Gamma^{[-,-]}(x, p_{\mathrm{T}}), \tag{34}$$

$$\Gamma_D^{[\mathrm{odd}]}(x, p_{\mathrm{T}}) = \frac{1}{2}\,\Gamma^{[+,-]}(x, p_{\mathrm{T}}) - \frac{1}{2}\,\Gamma^{[-,+]}(x, p_{\mathrm{T}}). \tag{35}$$

The two T-odd correlators reduce to the two three-gluon gluonic pole correlators $\Gamma_G^F(x,x)$ and $\Gamma_G^D(x,x)$, which differ in the way the three colour octets are coupled to a colour singlet.

The contributions $\delta\Phi$ and $\delta\Gamma$ make the nonuniversality explicit, which is the first step if one wants to study and possibly prove factorization in the case of TMD correlators. For phenomenological studies a reasonable first step is to omit the junk TMD, knowing that they will average to zero in a weighted asymmetry. This approximation can be applied immediately in the TMD master formula (eq. (17)) in which the colour flow possibilities are distinguished. This master formula remains the basic starting point, used in recent analyses of photon-jet [11] and jet-jet [12] production in hadron–hadron scattering. In the case of a linear weighting with the transverse momentum one can conveniently cast the result into folding of T-even and T-odd functions with normal and gluonic pole partonic cross-sections, respectively. The procedure has been investigated for several processes [13], in particular comparing the effect of using normal vs. gluonic pole cross-sections.

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